STATE-MORPHISM ALGEBRAS - GENERAL APPROACH

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Abstract. We present a complete description of subdirectly irreducible state BL-algebras as well as of subdirectly irreducible state-morphism BL-algebras. In addition, we present a general theory of state-morphism algebras, that is, algebras of general type with state-morphism which is an idempotent endomorphism. We define a diagonal state-morphism algebra and we show that every subdirectly irreducible state-morphism algebra can be embedded into a diagonal one. We describe generators of varieties of state-morphism algebras, in particular ones of state-morphism BL-algebras, state-morphism MTL-algebras, state-morphism non-associative BL-algebras, and state-morphism pseudo MV-algebras.

1. Introduction

A state, as an analogue of a probability measure, is a basic notion of the theory of quantum structures, see e.g. [14]. However, for MV-algebras, the state as averaging the truth value in the Łukasiewicz logic was introduced firstly by Mundici in [22], 40 years after introducing MV-algebras, [6]. We recall that a state on an MV-algebra $\mathbf{M}$ is a mapping $s : M \to [0,1]$ such that (i) $s(a \oplus b) = s(a) + s(b)$, if $a \odot b = 0$, and (ii) $s(1) = 1$. The property (i) says that s is additive on mutually excluding events a and b. It is important note that every non-degenerate MV-algebra admits at least one state. The set of states is a convex set, which in the weak topology of states is a compact Hausdorff set, and every extremal state is in fact an MV-algebra homomorphism from $\mathbf{M}$ into the MV-algebra of the real interval $[0,1]$, and vice-versa, [22]. In addition, extremal states generate the set of all states because by the Krein-Mil'man Theorem, [18, Thm 5.17], every state is a weak limit of a net of convex combinations of these special homomorphisms.

In the last decade, the states entered into theory of MV-algebras in a very ambitious manner. In [23, 21], authors have showed a relation between states

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and de Finetti's approach to probability in terms of bets. In addition, Panti and independently Kroupa in [24, 20] have showed that every state on $\mathbf{M}$ is an integral through a unique regular Borel probability measure concentrated on the set of extremal states on $\mathbf{M}$.

Nevertheless as we have seen states are not a proper notion of universal algebra, and therefore, they do not provide an algebraizable logic for probabilistic reasoning of the many-valued approach.

Recently, Flaminio and Montagna in [16] presented an algebraizable logic containing probabilistic reasoning, and its equivalent algebraic semantic is the variety of state MV-algebras. We recall that a *state MV-algebra* is an MV-algebra whose language is extended adding an operator, τ (called also an *internal state*), whose properties are inspired by the ones of states. The analogues of extremal states are *state-morphism operators*, introduced in [7]. By definition, it is an idempotent endomorphism on an MV-algebra.

State MV-algebras generalize, for example, Hájek's approach, [19], to fuzzy logic with modality Pr (interpreted as *probably*) which has the following semantic interpretation: The probability of an event a is presented as the truth value of $\mathrm{Pr}(a)$. On the other hand, if s is a state, then $s(a)$ is interpreted as averaging of appearing the many valued event a.

We note that if $(\mathbf{M},\tau)$ is a state MV-algebra, assuming that that the range $\tau(\mathbf{M})$ is simple, we see that it is a subalgebra of the real interval $[0,1]$ and therefore, τ can be regarded as a standard state on $\mathbf{M}$. On the other hand, every MV-algebra $\mathbf{M}$ can be embedded into the tensor product $[0,1]\otimes\mathbf{M}$, therefore, given a state s on $\mathbf{M}$, we define an operator τ_s on $[0,1]\otimes\mathbf{M}$ via $\tau_s(t\otimes a) := t\cdot s(a)$, [16, Thm 5.3]. Then due to [7, Thm 3.2], τ_s is a state-operator that is a state-morphism operator iff s is an extremal state. Thus, there is a natural correspondence between the notion of a state and an extremal state on one side, and a state-operator and a state-morphism operator on the other side.

Subdirectly irreducible state-morphism MV-algebras were described in [7, 9] and this was extended also for state-morphism BL-algebras in [11]. A complete description of both subdirectly irreducible state MV-algebras as well as subdirectly irreducible state-morphism MV-algebras can be found in [13]. In [8], it was shown that if $(\mathbf{M},\tau)$ is a state MV-algebra whose image $\tau(\mathbf{M})$ belongs to the variety generated by the $L_1,\dots,L_n$, where $L_i := \{0,1/i,\dots,i/i\}$, then τ has to be a state-morphism operator. The same is true if $\mathbf{M}$ is linearly ordered, [7]. Recently, in [13], we have shown that the unit square $[0,1]^2$ with the diagonal operator generates the whole variety of state-morphism MV-algebras; it answered in positive an open problem posed in [7]. In addition, there was shown that in contrast to MV-algebras, the lattice of subvarieties is uncountable. Moreover, it was shown that every subdirectly irreducible state-morphism MV-algebra can be embedded into some diagonal one.

In this paper, we continue in the study of state BL-algebras and state-morphism BL-algebras. Because the methods developed in [13] are so general that, it is possible to study more general structures than MV-algebras or BL-algebras under a common umbrella. Hence, we introduce state-morphism algebras $(\mathbf{A},\tau)$, where the algebra $\mathbf{A}$ is an arbitrary algebra of type F and τ is an idempotent endomorphism of $\mathbf{A}$. Then general results applied to special types of algebras give interesting new results.

The main goals of the paper are:

(1) Complete characterizations of subdirectly irreducible state BL-algebras and state-morphism BL-algebras.

(2) Showing that every subdirectly state-morphism algebra can be embedded into some diagonal one $D(\mathbf{B}) := (\mathbf{B}\times\mathbf{B}, \tau_B)$, where $\tau(a,b) = (a,a)$, $a, b \in B$, which is also subdirectly irreducible.

(3) We show that if $\mathcal{K}$ is a generator of some variety $\mathcal{V}$ of algebras of type F, then the system of diagonal state-morphism algebras $\{D(\mathbf{B}) : \mathbf{B} \in \mathcal{K}\}$ is a generator of the variety of state-morphism algebras whose F-reduct belongs to $\mathcal{V}$.

(4) We exhibit cases when the Congruence Extension Property holds for a variety of state-morphism algebras.

(5) In particular, a generator of the variety of state-morphism BL-algebras is the class of all BL-algebras of the real interval $[0,1]$ equipped with a continuous t-norm. Similarly, a generator of the variety of state-morphism MTL-algebras is the class of all MTL-algebras of the real interval equipped with a left-continuous t-norm, similarly for non-associative BL-algebras one is the set of all non-associative BL-algebras of the real interval $[0,1]$ equipped with a non-associative t-norm, and a generator of the variety of state-morphism pseudo MV-algebras is any pseudo MV-algebra $\Gamma(G,u)$, where (G,u) is a doubly transitive unital ℓ-group.

2. Subdirectly Irreducible State BL-algebras

In this section, we define state BL-algebras and state-morphism BL-algebras and we present a complete description of their subdirectly irreducible algebras. These results generalize those from [7, 9, 11, 13].

We recall that according to [19], a *BL-algebra* is an algebra $\mathbf{M} = (M; \wedge, \vee, \odot, \rightarrow, 0, 1)$ of the type $\langle 2,2,2,2,0,0\rangle$ such that $(M; \wedge, \vee, 0, 1)$ is a bounded lattice, $(M; \odot, 1)$ is a commutative monoid, and for all $a, b, c \in M$,

(1) $c \le a \to b$ iff $a \odot c \le b$;
(2) $a \wedge b = a \odot (a \to b)$;
(3) $(a \to b) \vee (b \to a) = 1$.

For any $a \in M$, we define a complement $a^- := a \to 0$. Then $a \le a^{--}$ for any $a \in M$ and a BL-algebra is an MV-algebra iff $a^{--} = a$ for any $a \in M$.

A non-empty set $F \subseteq M$ is called a *filter* of $\mathbf{M}$ (or a *BL-filter* of $\mathbf{M}$) if for every $x, y \in M$: (1) $x, y \in F$ implies $x \odot y \in F$, and (2) $x \in F$, $x \le y$ implies $y \in F$. A filter $F \neq M$ is called a *maximal filter* if it is not strictly contained in any other filter $F' \neq M$. A BL-algebra is called *local* if it has a unique maximal filter.

We denote by $\mathrm{Rad}_1(\mathbf{M})$ the intersection of all maximal filters of $\mathbf{M}$.

Let $\mathbf{M}$ be a BL-algebra. A mapping $\tau : M \to M$ such that, for all $x, y \in M$, we have

$(1)_{BL}$ $\tau(0) = 0$;
$(2)_{BL}$ $\tau(x \to y) = \tau(x) \to \tau(x \wedge y)$;
$(3)_{BL}$ $\tau(x \odot y) = \tau(x) \odot \tau(x \to (x \odot y))$;
$(4)_{BL}$ $\tau(\tau(x) \odot \tau(y)) = \tau(x) \odot \tau(y)$;
$(5)_{BL}$ $\tau(\tau(x) \to \tau(y)) = \tau(x) \to \tau(y)$

is said to be a *state-operator* on $\mathbf{M}$, and the pair $(\mathbf{M}, \tau)$ is said to be a *state BL-algebra*, or more precisely, a *BL-algebra with internal state*.

If $\tau : M \to M$ is a BL-endomorphism such that $\tau \circ \tau = \tau$, then τ is a state-operator on $\mathbf{M}$ and it is said to be a *state-morphism operator* and the couple $(\mathbf{M}, \tau)$ is said to be a *state-morphism BL-algebra*.

A filter F of a BL-algebra $\mathbf{M}$ is said to be a τ-filter if $\tau(F) \subseteq F$. If τ is a state-operator on $\mathbf{M}$, we denote by

$$\mathrm{Ker}(\tau) = \{a \in M : \tau(a) = 1\}.$$

then $\mathrm{Ker}(\tau)$ is a τ-filter. A state-operator τ is said to be *faithful* if $\mathrm{Ker}(\tau) = \{1\}$.

We recall that there is a one-to-one relation between congruences and τ-filters on a state BL-algebra $(\mathbf{M}, \tau)$ as follows. If F is a τ-filter, then the relation $\sim_F$ given by $x \sim_F y$ iff $x \to y, y \to x \in F$ is a congruence of the BL-algebra $\mathbf{M}$ and $\sim_F$ is also a congruence of the state BL-algebra $(\mathbf{M}, \tau)$.

Conversely, let $\sim$ be a congruence of state BL-algebra $(\mathbf{M}, \tau)$ and set $F_\sim := \{x \in M : x \sim 1\}$. Then $F_\sim$ is a τ-filter of $(\mathbf{M}, \tau)$ and $\sim_{F_\sim} = \sim$ and $F = F_{\sim_F}$.

By [5, Lem 3.5(k)], $(\tau(\mathbf{M}), \tau)$ is a subalgebra of $(\mathbf{M}, \tau)$, τ on $\tau(M)$ is the identity, and hence, $(\mathrm{Ker}(\tau); \to, 0, 1)$ is a subhoop of $\mathbf{M}$. We say that two subhoops, A and B, of a BL-algebra $\mathbf{M}$ have the *disjunction property* if for all $x \in A$ and $y \in B$, if $x \vee y = 1$, then either $x = 1$ or $y = 1$.

Nevertheless a subdirectly irreducible state BL-algebra $(\mathbf{M}, \tau)$ is not necessarily linearly ordered, according to [5, Thm 5.5], $\tau(\mathbf{M})$ is always linearly ordered.

We note that according to [5, Prop 3.13], if $\mathbf{M}$ is an MV-algebra, then the notion of a state MV-algebra coincides with the notion of a state BL-algebra.

The following three characterizations were originally proved in [13] for state MV-algebras. Here we show that the original proofs from [13] slightly improved work also for state BL-algebras.

Lemma 2.1. *Suppose that* $(\mathbf{M}, \tau)$ *is a state BL-algebra. Then:*

(1) *If* τ *is faithful, then* $(\mathbf{M}, \tau)$ *is a subdirectly irreducible state BL-algebra if and only if* $\tau(\mathbf{M})$ *is a subdirectly irreducible BL-algebra.*

Now let $(\mathbf{M}, \tau)$ *be subdirectly irreducible. Then:*

(2) $\mathrm{Ker}(\tau)$ *is (either trivial or) a subdirectly irreducible hoop.*
(3) $\mathrm{Ker}(\tau)$ *and* $\tau(\mathbf{M})$ *have the disjunction property.*

Proof. (1) Suppose τ is faithful. Let F denote the least nontrivial τ-filter of $(\mathbf{M}, \tau)$. There are two cases: (i) If $\tau(M) \cap F \neq \{1\}$, then $\tau(M) \cap F$ is the least nontrivial filter of $\tau(\mathbf{M})$ and $\tau(\mathbf{M})$ is subdirectly irreducible. (ii) If $\tau(\mathbf{M}) \cap F = \{1\}$, then for all $x \in F$, $\tau(x) = 1$ because $\tau(x) \in \tau(M) \cap F$ and $F \subseteq \mathrm{Ker}(\tau) = \{1\}$ is the trivial filter, a contradiction. Therefore, only the first case is possible and $\tau(\mathbf{M})$ is subdirectly irreducible.

Conversely, let $\tau(\mathbf{M})$ be subdirectly irreducible and let G be the least nontrivial filter of $\tau(\mathbf{M})$. Then the τ-filter F of $(\mathbf{M}, \tau)$ generated by G is the least nontrivial τ-filter of $(\mathbf{M}, \tau)$. Indeed, if K is another nontrivial τ-filter of $(\mathbf{M}, \tau)$, then $K \cap \tau(M) \supseteq F \cap \tau(M) = G$. Then K contains the τ-filter generated by G, that is $F \subseteq K$ which proves F is the least and $(\mathbf{M}, \tau)$ is subdirectly irreducible.

Now let $(\mathbf{M}, \tau)$ be subdirectly irreducible and let F denote the least nontrivial filter of $(\mathbf{M}, \tau)$.

(2) Suppose that τ is not faithful. Then $\mathrm{Ker}(\tau)$ is a nontrivial τ-filter. If $(\mathbf{M}, \tau)$ is subdirectly irreducible, it has a least nontrivial τ-filter, F say. So, $F \subseteq \mathrm{Ker}(\tau)$,

and hence F is the least nontrivial filter of the hoop $\mathrm{Ker}(\tau)$. Hence, $\mathrm{Ker}(\tau)$ is a subdirectly irreducible hoop.

(3) Suppose, by way of contradiction, that for some $x \in \mathrm{Ker}(\tau)$ and $y = \tau(y) \in \tau(M)$ one has $x < 1$, $y < 1$ and $x \vee y = 1$. It is easy to see that the BL-filters generated by x and by y, respectively, are τ-filters. Therefore they both contain F. Hence, the intersection of these filters contains F. Now let $c < 1$ be in F. Then there is a natural number n such that $x^n \leq c$ and $y^n \leq c$. It follows that $1 = (x \vee y)^n = x^n \vee y^n \leq c$, a contradiction. □

Lemma 2.2. *If* $(\mathbf{M}, \tau)$ *is a subdirectly irreducible state BL-algebra, then* $\tau(M)$ *and* $\mathrm{Ker}(\tau)$ *are linearly ordered.*

Proof. According to [5, Thm 5.5], $\tau(M)$ is always linearly ordered. On the other hand, by Lemma 2.1, $\mathrm{Ker}(\tau)$ is either a trivial hoop or a subdirectly irreducible hoop, and hence it is linearly ordered. □

Theorem 2.3. *Let* $(\mathbf{M}, \tau)$ *be a state BL-algebra satisfying conditions* (1), (2) *and* (3) *in Lemma* 2.1*. Then* $(\mathbf{M}, \tau)$ *is subdirectly irreducible.*

Proof. Suppose first that τ is faithful and that $\tau(\mathbf{M})$ is subdirectly irreducible. Let F_0 be the least nontrivial filter of $\tau(\mathbf{M})$ and let F be the BL-filter of $\mathbf{M}$ generated by F_0. Then F is a τ-filter. Indeed, if $x \in F$, then there is $\tau(a) \in F_0$ and a natural number n such that $\tau(a)^n \leq x$. It follows that $\tau(x) \geq \tau(\tau(a)^n) = \tau(a)^n$, and $\tau(x) \in F$.

We assert that F is the least nontrivial τ-filter of $(\mathbf{M}, \tau)$. First of all, $\tau(\mathbf{M})$, being a subdirectly irreducible BL-algebra, is linearly ordered. Now in order to prove that F is the least nontrivial τ-filter of $(\mathbf{M}, \tau)$, it suffices to prove that every τ-filter G not containing F is trivial. Now let $c < 1$ in $F \backslash G$. Then since $\mathrm{Ker}(\tau) = \{1\}$, $\tau(c) < 1$. Next, let $d \in G$. Then $\tau(d) \in G$, and for every n it cannot be $\tau(d)^n \leq \tau(c)$, otherwise $\tau(c) \in G$. Hence, for every n, $\tau(c) < \tau(d)^n$, and hence $\tau(c)$ does not belong to the τ-filter of $\tau(\mathbf{M})$ generated by $\tau(d)$. By the minimality of F in $\tau(\mathbf{M})$, $\tau(d) = 1$ and since τ is faithful, we conclude that $d = 1$ and G is trivial, as desired.

Now suppose that $\mathrm{Ker}(\tau)$ is nontrivial. By condition (2), $\mathrm{Ker}(\tau)$ is subdirectly irreducible. Thus, let F be the least nontrivial filter of $\mathrm{Ker}(\tau)$. Then F is a non trivial τ-filter, and we have to prove that F is the least nontrivial τ-filter of $(\mathbf{M}, \tau)$. Let G be any non trivial τ-filter of $(\mathbf{M}, \tau)$. If $G \subseteq \mathrm{Ker}(\tau)$, then it contains the least filter, F, of $\mathrm{Ker}(\tau)$, and $F \subseteq G$. Otherwise, G contains some $x \notin \mathrm{Ker}(\tau)$, and hence it contains $\tau(x) < 1$. Now by the disjunction property, for all $y < 1$ in $\mathrm{Ker}(\tau)$, $\tau(x) \vee y < 1$ and $\tau(x) \vee y \in \mathrm{Ker}(\tau) \cap G$. Thus, G contains the filter generated by $\tau(x) \vee y$, which is a non trivial filter of the hoop $\mathrm{Ker}(\tau)$, and hence it contains F, the least nontrivial filter of $\mathrm{Ker}(\tau)$. This proves the claim. □

By [13, Thm 3.5], conditions (1), (2), and (3) from Lemma 2.1 are independent ones even for state BL-algebras. In addition, Theorem 2.3 gives a characterization of subdirectly irreducible state BL-algebras. If $(\mathbf{M}, \tau)$ is a state-morphism BL-algebra, combining [11, Thm 4.5] we can say more about subdirectly irreducible state-morphism BL-algebras. The following examples are from [11].

Example 2.4. Let $\mathbf{M}$ be a BL-algebra. On $M \times M$ we define two operators, τ_1 and τ_2, as follows

$$\tau_1(a, b) = (a, a), \quad \tau_2(a, b) = (b, b), \quad (a, b) \in M \times M. \tag{2.0}$$

Then τ_1 and τ_2 are two state-morphism operators on $\mathbf{M}\times\mathbf{M}$. Moreover, $(\mathbf{M}\times\mathbf{M},\tau_1)$ and $(\mathbf{M}\times\mathbf{M},\tau_2)$ are isomorphic state BL-algebras under the isomorphism $(a,b)\mapsto(b,a)$.

We say that an element $a\in M$ is *Boolean* if $a^{--}=a$ and $a\odot a=a$. Let $B(\mathbf{M})$ be the set of Boolean elements. Then $0,1\in B(\mathbf{M})$, $B(\mathbf{M})$ is a subset of the MV-skeleton $\mathrm{MV}(\mathbf{M}):=\{x\in M : x^{--}=x\}$, and $a\in B(\mathbf{M})$ implies $a^-\in B(\mathbf{M})$. We recall that according to [26, Thm 2], $\mathrm{MV}(\mathbf{M})$ is an MV-algebra, therefore, $B(\mathbf{M})$ is a Boolean subalgebra of $\mathrm{MV}(\mathbf{M})$.

Example 2.5. Let $\mathbf{B}$ be a local MV-algebra such that $\mathrm{Rad}_1(\mathbf{B})\neq\{1\}$ is a unique nontrivial filter of B. Let $\mathbf{M}$ be a subalgebra of $\mathbf{B}\times\mathbf{B}$ that is generated by $\mathrm{Rad}_1(\mathbf{B})\times\mathrm{Rad}_1(\mathbf{B})$, that is $M=(\mathrm{Rad}_1(\mathbf{B})\times\mathrm{Rad}_1(\mathbf{B}))\cup(\mathrm{Rad}_1(\mathbf{B})\times\mathrm{Rad}_1(\mathbf{B}))^-$. Let $\tau(x,y):=(x,x)$ for all $x,y\in M$. Then τ is a state-morphism operator on $\mathbf{M}$, $\mathrm{Ker}(\tau)=\{1\}\times\mathrm{Rad}_1(\mathbf{B})\subset\mathrm{Rad}_1(\mathbf{M})=\mathrm{Rad}_1(\mathbf{B})\times\mathrm{Rad}_1(\mathbf{B})$, $\mathbf{M}$ has no Boolean nontrivial elements, and $(\mathbf{M},\tau)$ is a subdirectly irreducible state-morphism MV-algebra that is not linear.

Example 2.6. Let $\mathbf{A}$ be a linear nontrivial BL-algebra and $\mathbf{B}$ a nontrivial subdirectly irreducible BL-algebra with the smallest nontrivial BL-filter F_B and let $h:\ \mathbf{A}\to\mathbf{B}$ be a BL-homomorphism. On $M=\mathbf{A}\times\mathbf{B}$ we define a mapping $\tau_h:\ M\to M$ by

$$\tau_h(a,b)=(a,h(a)),\quad (a,b)\in M. \tag{2.2}$$

If we set $y=(0,1)$ and $y^-=(1,0)$, then y and y^- are unique nontrivial Boolean elements.

Then τ_h is a state-morphism operator on $\mathbf{M}$ and $(\mathbf{M},\tau_h)$ is a subdirectly irreducible state-morphism BL-algebra iff $\mathrm{Ker}(h)=\{a\in A : h(a)=1\}=\{1\}$. In such a case, $\mathrm{Ker}(\tau_h)=\{1\}\times B$ and $F:=\{1\}\times F_B$ is the least nontrivial state-morphism filter on $\mathbf{M}$.

Now we present the main result on the complete characterization of subdirectly irreducible state-morphism BL-algebras which is a combination of [11, Thm 4.5] and Theorem 2.3.

Theorem 2.7. *A state-morphism BL-algebra $(\mathbf{M},\tau)$ is subdirectly irreducible if and only if one of the following three possibilities holds.*

(i) *$\mathbf{M}$ is linear, $\tau=\mathrm{Id}_M$ is the identity on M, and the BL-reduct $\mathbf{M}$ is a subdirectly irreducible BL-algebra.*

(ii) *The state-morphism operator τ is not faithful, $\mathbf{M}$ has no nontrivial Boolean elements, and the BL-reduct $\mathbf{M}$ of $(\mathbf{M},\tau)$ is a local BL-algebra, $\mathrm{Ker}(\tau)$ is a subdirectly irreducible irreducible hoop, and $\mathrm{Ker}(\tau)$ and $\tau(\mathbf{M})$ have the disjunction property.*

Moreover, $\mathbf{M}$ is linearly ordered if and only if $\mathrm{Rad}_1(\mathbf{M})$ is linearly ordered, and in such a case, $\mathbf{M}$ is a subdirectly irreducible BL-algebra such that if F is the smallest nontrivial state-filter for $(\mathbf{M},\tau)$, then F is the smallest nontrivial BL-filter for $\mathbf{M}$.

If $\mathrm{Rad}_1(\mathbf{M})=\mathrm{Ker}(\tau)$, then $\mathbf{M}$ is linearly ordered.

(iii) *The state-morphism operator τ is not faithful, $\mathbf{M}$ has a nontrivial Boolean element. There are a linearly ordered BL-algebra $\mathbf{A}$, a subdirectly irreducible BL-algebra $\mathbf{B}$, and an injective BL-homomorphism $h:\ \mathbf{A}\to\mathbf{B}$ such that $(\mathbf{M},\tau)$ is isomorphic as a state-morphism BL-algebra with the*

state-morphism BL-algebra $(\mathbf{A}\times\mathbf{B},\tau_h)$, *where* $\tau_h(x,y)=(x,h(x))$ *for any* $(x,y)\in A\times B$.

Proof. It follows from [11, Thm 4.5] and Theorem 2.3. $\square$

We recall that a *t-norm* is a function $t:[0,1]\times[0,1]\to[0,1]$ such that (i) t is commutative, associative, (ii) $t(x,1)=x$, $x\in[0,1]$, and (iii) t is nondecreasing in both components. If t is continuous, we define $x\odot_t y=t(x,y)$ and $x\to_t y=\sup\{z\in[0,1]:t(z,x)\le y\}$ for $x,y\in[0,1]$, then $\mathbb{I}_t:=([0,1];\min,\max,\odot_t,\to_t,0,1)$ is a BL-algebra. Moreover, according to [3, Thm 5.2], the variety of all BL-algebras is generated by all $\mathbb{I}_t$ with a continuous t-norm t. Let $\mathcal{T}$ denote the system of all BL-algebras $\mathbb{I}_t$, where t is any continuous t-norm.

The proof of the following result will follow from Theorem 5.2.

Theorem 2.8. *The variety of all state-morphism BL-algebras is generated by the system* $\{D(\mathbb{I}_t):t\in\mathcal{T}\}$.

3. General State-Morphism Algebras

In this section, we generalize the notion of state-morphism BL-algebras to an arbitrary variety of algebras of some type. It is interesting that many results known only for state-morphism MV-algebras or state-morphism BL-algebras have a very general presentation as state-morphism algebras. The main result of this section, Theorem 3.7, says that every subdirectly irreducible state-morphism algebra can be embedded into some diagonal one.

Let $\mathbf{A}$ be any algebra of type F and let $\operatorname{Con}\mathbf{A}$ be the system of congruences on $\mathbf{A}$ with the least congruence $\Delta_{\mathbf{A}}$. An endomorphisms $\tau:\mathbf{A}\longrightarrow\mathbf{A}$ satisfying $\tau\circ\tau=\tau$ is said to be a *state-morphism* on $\mathbf{A}$ and a couple $(\mathbf{A},\tau)$ is said to be a *state-morphism algebra* or an algebra with internal state-morphism. Clearly, if $\mathcal{K}$ is a variety of algebras of type F, then the class $\mathcal{K}_\tau$ of all state-morphism algebras $(\mathbf{A},\tau)$, where $\mathbf{A}\in\mathcal{K}$ and τ is any state-morphism on $\mathbf{A}$, forms a variety, too.

In the rest of the paper, we will assume that $\mathbf{A}$ is an arbitrary algebra with a fixed type F; if $\mathbf{A}$ is of a specific type, it will be said that and specified.

Definition 3.1. Let $\mathbf{B}\in\mathcal{K}$. Then an algebra $D(\mathbf{D}):=(\mathbf{B}\times\mathbf{B},\tau_B)$, where τ_B is defined by $\tau_B(x,y)=(x,x)$, $x,y\in B$, is a state-morphism algebra (more precisely $(\mathbf{B}\times\mathbf{B},\tau_B)\in\mathcal{K}_\tau$); we call τ_B also a *diagonal state-operator.* If a state-morphism algebra $(\mathbf{C},\tau)$ can be embedded into some diagonal state-morphism algebra, $(\mathbf{B}\times\mathbf{B},\tau_B)$, $(\mathbf{C},\tau)$ is said to be a *subdiagonal* state-morphism algebra*, or, more precisely,* $\mathbf{B}$-*subdiagonal.*

Let $(\mathbf{A},\tau)$ be a state-morphism algebra. We introduce the following sets:

$$\theta_\tau=\{(x,y)\in A\times A:\tau(x)=\tau(y)\},\tag{3.1}$$

$$\tau(A)=\{\tau(x):x\in A\}.$$

The subalgebra which is an image of $\mathbf{A}$ by τ is denoted by $\tau(\mathbf{A})$ and thus $\tau(\mathbf{A})\in\mathcal{K}$ and $(\tau(\mathbf{A}),\mathrm{Id}_{\tau(A)})\in\mathcal{K}_\tau$, where $\mathrm{Id}_{\tau(A)}$ is the identity on $\tau(A)$; we have also $\tau|\tau(A)=\mathrm{Id}_{\tau(A)}$.

If $\phi\in\operatorname{Con}\tau(\mathbf{A})$, we define

$$\theta_\phi:=\{(x,y)\in A\times A:(\tau(x),\tau(y))\in\phi\}.\tag{3.2}$$

Finally, if $\phi \subseteq A^2$ then the congruence on $\mathbf{A}$ generated by ϕ is denoted by $\Theta(\phi)$ and the congruence on $(\mathbf{A},\tau)$ generated by ϕ is denoted by $\Theta_\tau(\phi)$. Clearly $\mathrm{Con}\,(\mathbf{A},\tau) \subseteq \mathrm{Con}\,\mathbf{A}$ and also $\Theta(\phi) \subseteq \Theta_\tau(\phi)$.

Lemma 3.2. *Let $(\mathbf{A},\tau)$ be a state-morphism algebra. For any $\phi \in \mathrm{Con}\,\tau(\mathbf{A})$, we have $\theta_\phi \in \mathrm{Con}\,(\mathbf{A},\tau)$, and $\theta_\phi \cap \tau(A)^2 = \phi$. In addition, $\theta_\tau \in \mathrm{Con}\,(\mathbf{A},\tau)$, $\phi \subseteq \theta_\phi$, and $\Theta_\tau(\phi) \subseteq \theta_\phi$.*

Proof. Clearly, θ_ϕ is reflexive and symmetric. Moreover, if $(x,y),(y,z) \in \theta_\phi$, then $\big(\tau(x),\tau(y)\big),\big(\tau(y),\tau(z)\big) \in \phi$ and thus $\big(\tau(x),\tau(z)\big) \in \phi$ which gives $(x,z) \in \theta_\phi$.

Let $f^{\mathbf{A}}$ be an n-ary operation on $\mathbf{A}$ and let $x_1,\dots,x_n,y_1,\dots,y_n \in A$ be such that $(x_i,y_i) \in \theta_\phi$ for any $i=1,\dots,n$. Then $\big(\tau(x_i),\tau(y_i)\big) \in \phi$ holds for any $i=1,\dots,n$. Due to $\phi \in \mathrm{Con}\,\tau(\mathbf{A})$, we obtain $\big(f^{\tau(\mathbf{A})}(\tau(x_1),\dots,\tau(x_n)), f^{\tau(\mathbf{A})}(\tau(y_1),\dots,\tau(y_n))\big) \in \phi$.

Because τ is an endomorphism, $\tau(f^{\mathbf{A}}(x_1,\dots,x_n)) = f^{\tau(\mathbf{A})}(\tau(x_1),\dots,\tau(x_n))$ and $\tau(f^{\mathbf{A}}(y_1,\dots,y_n)) = f^{\tau(\mathbf{A})}(\tau(y_1),\dots,\tau(y_n))$ which gives $\big(\tau(f^{\mathbf{A}}(x_1,\dots,x_n)), \tau(f^{\mathbf{A}}(y_1,\dots,y_n))\big) \in \phi$ and finally also $\big(f^{\mathbf{A}}(x_1,\dots,x_n), f^{\mathbf{A}}(y_1,\dots,y_n)\big) \in \theta_\phi$.

Moreover, take an arbitrary $(x,y) \in \theta_\phi$. Then $\big(\tau(\tau(x)),\tau(\tau(y))\big) = \big(\tau(x),\tau(y)\big) \in \phi$ which gives $\big(\tau(x),\tau(y)\big) \in \theta_\phi$.

Hence, $\theta_\phi \in \mathrm{Con}\,(\mathbf{A},\tau)$ and if $\phi = \Delta_{\tau(\mathbf{A})}$, then $\theta_\phi = \theta_\tau$.

It is clear that $\theta_\phi \cap \tau(A)^2 \supseteq \phi$. Now let $(x,y) \in \theta_\phi \cap \tau(A)^2$. Then $x,y \in \tau(A)$, $(\tau(x),\tau(y)) \in \phi \subseteq \tau(A)^2$, so that $x=\tau(x) \in \tau(A)$, $y = \tau(y) \in \tau(A)$, and consequently, $(x,y) \in \phi$.

It is evident that θ_τ is a congruence on $(\mathbf{A},\tau)$.

Finally, if $(x,y) \in \phi$ then $\tau(x) = x$ and $\tau(y) = y$ which gives $\big(\tau(x),\tau(y)\big) = (x,y) \in \phi$. Thus $(x,y) \in \theta_\phi$ which finishes the proof that $\phi \subseteq \theta_\phi$ and $\Theta_\tau(\phi) \subseteq \theta_\phi$. ☐

Lemma 3.3. *Let $\theta \in \mathrm{Con}\,\mathbf{A}$ be such that $\theta \subseteq \theta_\tau$. Then $\theta \in \mathrm{Con}\,(\mathbf{A},\tau)$ holds.*

Moreover, if $x,y \in A$ are such that $(x,y) \in \theta_\tau$, then $\Theta(x,y) = \Theta_\tau(x,y)$.

Proof. If $(x,y) \in \theta \subseteq \theta_\tau$, then $\tau(x) = \tau(y)$ and thus $\big(\tau(x),\tau(y)\big) = \big(\tau(x),\tau(x)\big) \in \theta$ proves that $\theta \in \mathrm{Con}\,(\mathbf{A},\tau)$.

Moreover, if $(x,y) \in \theta_\tau$, then $\Theta(x,y) \subseteq \theta_\tau$. Due to the first part of Lemma, we obtain $\Theta(x,y) \in \mathrm{Con}\,(\mathbf{A},\tau)$ and thus $\Theta_\tau(x,y) \subseteq \Theta(x,y)$ holds. The second inclusion is trivial. ☐

Lemma 3.4. *If $x,y \in \tau(\mathbf{A})$, then $\Theta(x,y) = \Theta_\tau(x,y)$. Consequently, $\Theta(\phi) = \Theta_\tau(\phi)$ whenever $\phi \subseteq \tau(A)^2$.*

Proof. Let us denote by ϕ the congruence on $\tau(\mathbf{A})$ generated by (x,y). Clearly we obtain the chain of inclusions $\phi \subseteq \Theta(x,y) \subseteq \Theta(\phi) \subseteq \theta_\phi$ (because $(x,y) \in \phi$ and $\phi \subseteq \theta_\phi$, see Lemma 3.2).

Assume $(a,b) \in \Theta(x,y)$, then $(a,b) \in \theta_\phi$ and thus $(\tau(a),\tau(b)) \in \phi \subseteq \Theta(x,y)$. Thus $\Theta(x,y) \in \mathrm{Con}\,(\mathbf{A},\tau)$ and $\Theta_\tau(x,y) \subseteq \Theta(x,y)$ holds. The second inclusion is trivial.

Finally, let $\phi \subseteq \tau(A)^2$. By [2, Thm 5.3], the both congruence lattices of $\mathbf{A}$ and $(\mathbf{A},\tau)$ are complete sublattices of the lattice of equivalencies on $\mathbf{A}$, and therefore, they have the same infinite suprema. Hence, by the first part of the lemma,

$$\Theta(\phi) = \bigvee_{(x,y)\in\phi} \Theta(x,y) = \bigvee_{(x,y)\in\phi} \Theta_\tau(x,y) = \Theta_\tau(\phi).$$

□

Remark 3.5. By Lemma 3.2, if ϕ is a congruence on $\tau(\mathbf{A})$, then θ_ϕ is an extension of ϕ on $(\mathbf{A},\tau)$ and $\Theta(\phi)=\Theta_\tau(\phi)\subseteq\theta_\phi$. There is a natural question whether $\Theta(\phi)=\theta_\phi$? The answer is positive if and only if τ is the identity on A. Indeed, if τ is the identity on A, the statement is evident, in the opposite case, we have $\theta_{\Delta_{\tau(\mathbf{A})}}=\theta_\tau\neq\Delta_{\mathbf{A}}=\Theta(\Delta_{\tau(\mathbf{A})})$.

Theorem 3.6. *Let $(\mathbf{A},\tau)$ be a subdirectly irreducible state-morphism algebra such that $\mathbf{A}$ is subdirectly reducible. Then there is a subdirectly irreducible algebra $\mathbf{B}$ such that $(\mathbf{A},\tau)$ is $\mathbf{B}$-subdiagonal.*

Proof. First, if $\theta_\tau=\Delta_{\mathbf{A}}$, then for any $x\in A$, the equality $\tau(x)=x$ holds and thus $\mathrm{Con}\,\mathbf{A}=\mathrm{Con}\,(\mathbf{A},\tau)$ which is absurd because $\mathbf{A}$ is subdirectly irreducible and $(\mathbf{A},\tau)$ is not subdirectly irreducible.

The subdirect irreducibility of $(\mathbf{A},\tau)$ implies that there is a least proper congruence $\theta_{\min}\in\mathrm{Con}\,(\mathbf{A},\tau)$. Moreover, due to Lemma 3.3, the congruence $\theta_{\min}$ is also a least proper congruence θ on $\mathbf{A}$ with $\theta\subseteq\theta_\tau$ and thus $\theta_{\min}$ is an atom in $\mathrm{Con}\,\mathbf{A}$. Let us denote

$$\theta_\tau^\perp=\{\theta\in\mathrm{Con}\,\mathbf{A}:\theta\cap\theta_\tau=\Delta_{\mathbf{A}}\}.$$

First, we prove that there exists proper $\theta\in\theta_\tau^\perp$. The subdirect reducibility of $\mathbf{A}$ shows that there exists proper $\theta\in\mathrm{Con}\,\mathbf{A}$ with $\theta_{\min}\not\subseteq\theta$. Hence, $\theta_\tau\cap\theta=\Delta_{\mathbf{A}}$ holds (because if $\theta_\tau\cap\theta\neq\Delta_{\mathbf{A}}$, then $\theta_\tau\cap\theta$ is a proper congruence contained in θ_τ and minimality of $\theta_{\min}$ yields $\theta_{\min}\subseteq\theta\cap\theta_\tau\subseteq\theta$, which is absurd).

Moreover, let us have $\theta_n\in\theta_\tau^\perp$ for any $n\in\mathbb{N}$ with $\theta_n\subseteq\theta_{n+1}$, then clearly $\bigvee_{n\in\mathbb{N}}\theta_n=\bigcup_{n\in\mathbb{N}}\theta_n\in\theta_\tau^\perp$. Due to Zorn's Lemma, there is maximal $\theta^*\in\theta_\tau^\perp$.

We have proved that both θ_τ and θ^* are proper congruences on $\mathbf{A}$ with $\theta_\tau\cap\theta^*=\Delta_{\mathbf{A}}$. By the Birkhoff Theorem about subdirect reducibility, $\mathbf{A}$ is a subdirect product of two algebras $\mathbf{A}/\theta_\tau$ and $\mathbf{A}/\theta^*$ with an embedding $h:\mathbf{A}\longrightarrow\mathbf{A}/\theta_\tau\times\mathbf{A}/\theta^*$ defined by $h(x)=(x/\theta_\tau,x/\theta^*)$.

Now we define the mapping $\psi:A/\theta_\tau\longrightarrow A/\theta^*$ by $\psi(x/\theta_\tau)=\tau(x)/\theta^*$. Clearly ψ is well-defined because $x/\theta_\tau=y/\theta_\tau$ yields $\tau(x)=\tau(y)$ and thus $\psi(x/\theta_\tau)=\tau(x)/\theta^*=\tau(y)/\theta^*=\psi(y/\theta_\tau)$. Let us suppose that $\psi(x/\theta_\tau)=\psi(y/\theta_\tau)$. Then $\tau(x)/\theta^*=\tau(y)/\theta^*$ and $\big(\tau(x),\tau(y)\big)\in\theta^*$. Hence, $\Theta(\tau(x),\tau(y))\subseteq\theta^*$ holds. Finally, if $\tau(x)\neq\tau(y)$ (thus $\Theta(\tau(x),\tau(y))$ is a proper congruence), then $\tau(x),\tau(y)\in\tau(\mathbf{A})$ and Lemma 3.4 yields $\Theta(\tau(x),\tau(y))\in\mathrm{Con}\,(\mathbf{A},\tau)$ and thus $\theta_{\min}\subseteq\Theta(\tau(x),\tau(y))\subseteq\theta^*$ which is absurd ($\theta_{\min}\subseteq\theta_\tau\cap\theta^*=\Delta_{\mathbf{A}}$). Therefore, the mapping ψ is injective.

We shall prove that ψ is a homomorphism (and thus an embedding). If $f^{\mathbf{A}}$ is an n-ary operation and $x_1/\theta_\tau,\dots,x_n/\theta_\tau\in\mathbf{A}/\theta_\tau$, then

$$\begin{aligned}
\psi(f^{\mathbf{A}/\theta_\tau}(x_1/\theta_\tau,\dots,x_n/\theta_\tau)) &= \psi(f^{\mathbf{A}}(x_1,\dots,x_n)/\theta_\tau)\\
&= \tau(f^{\mathbf{A}}(x_1,\dots,x_n))/\theta^*\\
&= f^{\mathbf{A}}(\tau(x_1),\dots,\tau(x_n))/\theta^*\\
&= f^{\mathbf{A}/\theta^*}(\tau(x_1)/\theta^*,\dots,\tau(x_n)/\theta^*)\\
&= f^{\mathbf{A}/\theta^*}(\psi(x_1/\theta_\tau),\dots,\psi(x_n/\theta_\tau)).
\end{aligned}$$

Now we prove that $\mathbf{A}$ is $\mathbf{A}/\theta^*$-diagonal. Let $g:A\longrightarrow(A/\theta^*)^2$ be defined via $g(x)=(\psi(x/\theta_\tau),x/\theta^*)=(\tau(x)/\theta^*,x/\theta^*)$. Because the mapping g is the composition of two functions h and ψ which are embeddings, g is also an embedding of $\mathbf{A}$

into $(\mathbf{A}/\theta^*)^2$. Now we can compute:

$$\begin{aligned} g(\tau(x)) &= \big(\tau(\tau(x))/\theta^*, \tau(x)/\theta^*\big) \\ &= \big(\tau(x)/\theta^*, \tau(x)/\theta^*\big) \\ &= \tau_{\mathbf{A}/\theta^*}(\tau(x)/\theta^*, x/\theta^*) \\ &= \tau_{\mathbf{A}/\theta^*}(g(x)), \end{aligned}$$

where $\tau_{\mathbf{A}/\theta^*}$ is the diagonal state-morphism on the product $\mathbf{A}/\theta^* \times \mathbf{A}/\theta^*$. Therefore, $g : (\mathbf{A}, \tau) \longrightarrow (\mathbf{A}/\theta^* \times \mathbf{A}/\theta^*, \tau_{\mathbf{A}/\theta^*})$ is an embedding and $(\mathbf{A}, \tau)$ is $\mathbf{A}/\theta^*$-diagonal.

Finally, we prove the subdirect irreducibility of $\mathbf{A}/\theta^*$. Of course, $\theta_{\min} \cap \theta^* = \Delta_{\mathbf{A}}$ yields $\theta_{\min} \not\subseteq \theta^*$ and thus $\theta^* \subset \theta^* \vee \theta_{\min}$. Moreover, if $\theta^* \subset \theta$, from maximality of θ^* we obtain $\theta \cap \theta_\tau \neq \Delta_{\mathbf{A}}$ and thus $\theta_{\min} \subseteq \theta_\tau \cap \theta$. Finally, $\theta_{\min} \vee \theta^* \subseteq (\theta_\tau \cap \theta) \vee \theta^* \subseteq (\theta_\tau \cap \theta) \vee \theta = \theta$ holds. Hence, for any congruence $\theta \in \operatorname{Con} \mathbf{A}$, the inequality $\theta^* \subset \theta^* \cap \theta_{\min} \subseteq \theta$ holds. Due to the Birkhoff's Theorem and the Second Homomorphism Theorem, an algebra $\mathbf{A}/\theta^*$ is subdirectly irreducible. $\square$

Theorem 3.6 can be extended as follows.

Theorem 3.7. *For every subdirectly irreducible state-morphism algebra* $(\mathbf{A}, \tau)$, *there is a subdirectly irreducible algebra* $\mathbf{B}$ *such that* $(\mathbf{A}, \tau)$ *is* $\mathbf{B}$*-subdiagonal.*

Proof. There are two cases: (1) $(\mathbf{A}, \tau)$ and $\mathbf{A}$ are subdirectly irreducible, and (2) $(\mathbf{A}, \tau)$ is a subdirectly irreducible state-morphism algebra and $\mathbf{A}$ is a subdirectly reducible algebra

(1) Assume that $(\mathbf{A}, \tau)$ and $\mathbf{A}$ are subdirectly irreducible. Define two state-morphism algebras $(\tau(\mathbf{A}) \times \mathbf{A}, \tau_1)$ and $(\mathbf{A} \times \mathbf{A}, \tau_2)$, where $\tau_1(a, b) = (a, a)$, $(a, b) \in \tau(A) \times A$, and $\tau_2(a, b) = (a, a)$, $a, b \in A$. Then the first one is a subalgebra of the second one.

Define a mapping $\phi : A \to \tau(A) \times A$ defined by $\phi(a) = (\tau(a), a)$, $a \in A$. Then ϕ is injective because if $\phi(a) = \phi(b)$ then $(\tau(a), a) = (\tau(b), b)$ and $a = b$. We show that ϕ is a homomorphism. Let $f^{\mathbf{A}}$ be an n-ary operation on $\mathbf{A}$ and let $a_1, \ldots, a_n \in A$. Then

$$\begin{aligned} \phi(f^{\mathbf{A}}(a_1, \ldots, a_n)) &= \big(\tau(f^{\mathbf{A}}(a_1, \ldots, a_n)), f^{\mathbf{A}}(a_1, \ldots, a_n)\big) \\ &= \big(f^{\mathbf{A}}(\tau(a_1), \ldots, \tau(a_n)), f^{\mathbf{A}}(a_1, \ldots, a_n)\big) \\ &= f^{\tau(\mathbf{A}) \times \mathbf{A}}\big((\tau(a_1), a_1), \ldots, (\tau(a_n), a_n)\big) \\ &= f^{\tau(\mathbf{A}) \times \mathbf{A}}(\phi(a_1), \ldots, \phi(a_n)). \end{aligned}$$

Since $\phi : \mathbf{A} \to \tau(\mathbf{A}) \times \mathbf{A} \subseteq \mathbf{A} \times \mathbf{A}$, ϕ can be assumed also as an injective homomorphism from the state-morphism algebra $(\mathbf{A}, \tau)$ into the state-morphism algebra $D(\mathbf{B})$, where $\mathbf{B} := \mathbf{A}$ is a subdirectly irreducible algebra.

(2) This case was proved in Theorem 3.6. $\square$

For example, a state-morphism algebra $(\mathbf{A}, \mathrm{Id}_A)$, where Id_A is the identity on A, is subdirectly irreducible if and only if $\mathbf{A}$ is subdirectly irreducible. Therefore, $(\mathbf{A}, \mathrm{Id}_A)$ can be embedded into $(\mathbf{A} \times \mathbf{A}, \tau_A)$ under the mapping $a \mapsto (a, a)$, $a \in A$. Consequently, every subdirectly irreducible state-morphism algebra $(\mathbf{A}, \mathrm{Id}_A)$ is $\mathbf{A}$-subdiagonal with $\mathbf{A}$ subdirectly irreducible.

We note that in the same way as in [13, Lem 6.1], it is possible to show that the class of subdiagonal state-morphism algebras is closed under subalgebras and ultraproducts, and not closed under homomorphic images, see [13, Lem 6.6].

4. Varieties of State-Morphism Algebras and Their Generators

In this section, we study varieties of state-morphism algebras and their generators. It is interesting to note that some similar results proved for state-morphism MV-algebras in [13] can be obtained in an analogous way also for a general variety of algebras.

Let τ be a state-morphism operator on an algebra $\mathbf{A}$. We set

$$\mathrm{Ker}(\tau) := \{(x,y) \in A \times A : \tau(x) = \tau(y)\},$$

the *kernel* of τ. We say that τ is *faithful* if $\mathrm{Ker}(\tau) = \Delta_{\mathbf{A}}$. It is evident that τ is faithful iff $\tau(x) = x$ for each $x \in A$. In addition, τ is faithful iff τ is injective.

For every class $\mathcal{K}$ of same type algebras, we set $\mathsf{D}(\mathcal{K}) = \{D(\mathbf{A}) : \mathbf{A} \in \mathcal{K}\}$, where $D(\mathbf{A}) = (\mathbf{A} \times \mathbf{A}, \tau_A)$.

As usual, given a class $\mathcal{K}$ of algebras of the same type, $\mathsf{I}(\mathcal{K})$, $\mathsf{H}(\mathcal{K})$, $\mathsf{S}(\mathcal{K})$ and $\mathsf{P}(\mathcal{K})$ and $\mathsf{P_U}(\mathcal{K})$ will denote the class of isomorphic images, of homomorphic images, of subalgebras, of direct products and of ultraproducts of algebras from $\mathcal{K}$, respectively. Moreover, $\mathsf{V}(\mathcal{K})$ will denote the variety generated by $\mathcal{K}$.

Lemma 4.1. (1) *Let $\mathcal{K}$ be a class of algebras of the same type F. Then* $\mathsf{VD}(\mathcal{K}) \subseteq \mathsf{V}(\mathcal{K})_\tau$.
(2) *Let $\mathcal{V}$ be any variety. Then $\mathcal{V}_\tau = \mathsf{ISD}(\mathcal{V})$.*

Proof. (1) If $D(\mathbf{A}) \in \mathsf{D}(\mathcal{K})$ (thus $\mathbf{A} \in \mathcal{K}$), then the F-reduct of the algebra $D(\mathbf{A})$ is the algebra $\mathbf{A} \times \mathbf{A}$ which belongs to the variety $\mathsf{V}(\mathcal{K})$. Due to definition of $\mathsf{V}(\mathcal{K})_\tau$, we obtain also $D(\mathbf{A}) \in \mathsf{V}(\mathcal{K})_\tau$. We have proved that $\mathsf{D}(\mathcal{K}) \subseteq \mathsf{V}(\mathcal{K})_\tau$. Because $\mathsf{V}(\mathcal{K})_\tau$ is a variety then also $\mathsf{VD}(\mathcal{K}) \subseteq \mathsf{V}(\mathcal{K})_\tau$

(2) Let $(\mathbf{A}, \tau) \in \mathcal{V}_\tau$. As we have seen in the proof of Theorem 3.7, the map $\phi : a \mapsto (\tau(a), a)$ is an injective homomorphism of $(\mathbf{A}, \tau)$ into $D(\mathbf{A})$. Hence, ϕ is compatible with τ, and $(\mathbf{A}, \tau) \in \mathsf{ISD}(\mathcal{V})$. Conversely, the F-reduct of any algebra in $\mathsf{D}(\mathcal{V})$ is in $\mathcal{V}$, (being a direct product of algebras in $\mathcal{V}$), and hence the F-reduct of any member of $\mathsf{ISD}(\mathcal{V})$ is in $\mathsf{IS}(\mathcal{V}) = \mathcal{V}$. Hence, any member of $\mathsf{ISD}(\mathcal{V})$ is in $\mathcal{V}_\tau$. $\square$

Lemma 4.2. *Let $\mathcal{K}$ be a class of algebras of the same type F. Then:*
(1) $\mathsf{DH}(\mathcal{K}) \subseteq \mathsf{HD}(\mathcal{K})$.
(2) $\mathsf{DS}(\mathcal{K}) \subseteq \mathsf{ISD}(\mathcal{K})$.
(3) $\mathsf{DP}(\mathcal{K}) \subseteq \mathsf{IPD}(\mathcal{K})$.
(4) $\mathsf{VD}(\mathcal{K}) = \mathsf{ISD}(\mathsf{V}(\mathcal{K}))$.

Proof. (1) Let $D(\mathbf{C}) \in \mathsf{DH}(\mathcal{K})$. Then there are $\mathbf{A} \in \mathcal{K}$ and a homomorphism h from $\mathbf{A}$ onto $\mathbf{C}$. Let for all $a, b \in A$, $h^*(a,b) = (h(a), h(b))$. We claim that h^* is a homomorphism from $D(\mathbf{A})$ onto $D(\mathbf{C})$. That h^* is a homomorphism is clear. We verify that h^* is compatible with τ_A. We have $h^*(\tau_A(a,b)) = h^*(a,a) = (h(a), h(a)) = \tau_C(h(a), h(b)) = \tau_C(h^*(a,b))$. Finally, since h is onto, given $(c,d) \in C \times C$, there are $a, b \in A$ such that $h(a) = c$ and $h(b) = d$. Hence, $h^*(a,b) = (c,d)$, h^* is onto, and $D(\mathbf{C}) \in \mathsf{HD}(\mathcal{K})$.

(2) It is trivial.

(3) Let $\mathbf{A} = \prod_{i \in I}(\mathbf{A}_i) \in \mathsf{P}(\mathcal{K})$, where each $\mathbf{A}_i$ is in $\mathcal{K}$. Then the map

$$\Phi : \big((a_i : i \in I), (b_i : i \in I)\big) \mapsto \big((a_i, b_i) : i \in I\big)$$

is an isomorphism from $D(\mathbf{A})$ onto $\prod_{i \in I} D(\mathbf{A}_i)$. Indeed, it is clear that Φ is an F-isomorphism. Moreover, denoting the state-morphism of $\prod_{i \in I} D(\mathbf{A}_i)$ by τ^*, we

get:

$$\Phi\big(\tau_A\big((a_i : i \in I),(b_i : i \in I)\big)\big) = \Phi\big((a_i : i \in I),(a_i : i \in I)\big) =$$
$$= \big((a_i,a_i) : i \in I\big) = \big(\tau_{\mathbf{A}_i}(a_i,b_i) : i \in I\big) = \tau^*\big(\Phi\big((a_i : i \in I),(b_i : i \in I)\big)\big),$$

and hence Φ is an isomorphism.

(4) By (1), (2) and (3), $\mathsf{DV}(\mathcal{K}) = \mathsf{DHSP}(\mathcal{K}) \subseteq \mathsf{HSPD}(\mathcal{K}) = \mathsf{VD}(\mathcal{K})$, and hence $\mathsf{ISDV}(\mathcal{K}) \subseteq \mathsf{ISVD}(\mathcal{K}) = \mathsf{VD}(\mathcal{K})$. Conversely, by Lemma 4.1(1), $\mathsf{VD}(\mathcal{K}) \subseteq \mathsf{V}(\mathcal{K})_\tau$, and by Lemma 4.1(2), $\mathsf{V}(\mathcal{K})_\tau = \mathsf{ISDV}(\mathcal{K})$. This proves the claim. □

Theorem 4.3. (1) *For every class $\mathcal{K}$ of algebras of the same type F,* $\mathsf{V}(\mathsf{D}(\mathcal{K})) = \mathsf{V}(\mathcal{K})_\tau$.

(2) *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two classes of same type algebras. Then $\mathsf{V}(D(\mathcal{K}_1)) = \mathsf{V}(D(\mathcal{K}_2))$ if and only if $\mathsf{V}(\mathcal{K}_1) = \mathsf{V}(\mathcal{K}_2)$.*

Proof. (1) By Lemma 4.2(4), $\mathsf{VD}(\mathcal{K}) = \mathsf{ISD}(\mathsf{V}(\mathcal{K}))$. Moreover, by Lemma 4.1(2), $\mathsf{V}(\mathcal{K})_\tau = \mathsf{ISDV}(\mathcal{K})$. Hence, $\mathsf{V}(\mathsf{D}(\mathcal{K})) = \mathsf{V}(\mathcal{K})_\tau$.

(2) We have $\mathsf{V}(\mathsf{D}(\mathcal{K}_1)) = \mathsf{V}(\mathcal{K}_1)_\tau$ and $\mathsf{V}(\mathsf{D}(\mathcal{K}_2)) = \mathsf{V}(\mathcal{K}_2)_\tau$. Clearly, $\mathsf{V}(\mathcal{K}_1) = \mathsf{V}(\mathcal{K}_2)$ implies $\mathsf{V}(\mathcal{K}_1)_\tau = \mathsf{V}(\mathcal{K}_2)_\tau$, and hence $\mathsf{V}(\mathsf{D}(\mathcal{K}_1)) = \mathsf{V}(\mathsf{D}(\mathcal{K}_2))$. Conversely, $\mathsf{V}(\mathsf{D}(\mathcal{K}_1)) = \mathsf{V}(\mathsf{D}(\mathcal{K}_2))$ implies $\mathsf{V}(\mathcal{K}_1)_\tau = \mathsf{V}(\mathcal{K}_2)_\tau$. But any algebra $\mathbf{A} \in \mathsf{V}(\mathcal{K}_1)$ is the F-reduct of a state-morphism algebra in $\mathsf{V}(\mathcal{K}_1)_\tau$, namely of $(\mathbf{A}, \mathrm{Id}_A)$.

It follows that, if $\mathsf{V}(\mathcal{K}_1)_\tau = \mathsf{V}(\mathcal{K}_2)_\tau$, then the classes of F-reducts of $\mathsf{V}(\mathcal{K}_1)_\tau$ and of $\mathsf{V}(\mathcal{K}_2)_\tau$ coincide, and hence $\mathsf{V}(\mathcal{K}_1) = \mathsf{V}(\mathcal{K}_2)$. □

As a direct corollary of Theorem 4.3, we have:

Theorem 4.4. *If a system $\mathcal{K}$ of algebras of the same type F generates the whole variety $\mathcal{V}(F)$ of all algebras of type F, then the variety $\mathcal{V}(F)_\tau$ of all state-morphism algebras $(\mathbf{A},\tau)$, where $\mathbf{A} \in \mathcal{V}(F)$, is generated by the class $\{D(\mathbf{A}) : \mathbf{A} \in \mathcal{K}\}$.*

Some applications of the latter theorem for different varieties of algebras will be done in Section 5.

Theorem 4.5. *If $\mathbf{A}$ is a subdirectly irreducible algebra, then any state-morphism algebra $(\mathbf{A},\tau)$ is subdirectly irreducible.*

Proof. Let $\mathbf{A}$ be a subdirectly irreducible algebra and let τ be a state-morphism operator on $\mathbf{A}$. If τ is the identity on A, then $\mathrm{Con}\,\mathbf{A} = \mathrm{Con}\,(\mathbf{A},\tau)$ and, consequently, $(\mathbf{A},\tau)$ is subdirectly irreducible. If τ is not the identity on A, then θ_τ, defined by (3.1), is a nontrivial congruence on $\mathbf{A}$, and thus $\theta_{\min} \subseteq \theta_\tau$, where $\theta_{\min} \in \mathrm{Con}\,\mathbf{A}$ is the least nontrivial congruence. Hence, $\theta_{\min}$ belongs to the set $\mathrm{Con}\,(\mathbf{A},\tau)$, see Lemma 3.3. Therefore, $\mathrm{Con}\,(\mathbf{A},\tau) \subseteq \mathrm{Con}\,\mathbf{A}$ yields the subdirect irreducibility of the algebra $(\mathbf{A},\tau)$, more precisely, $\theta_{\min}$ is also the least proper congruence in $\mathrm{Con}\,(\mathbf{A},\tau)$. □

We remind the following Mal'cev Theorem, [2, Lem 3.1].

Theorem 4.6. *Let $\mathbf{A}$ be an algebra and $\phi \subseteq A^2$. Then $(a,b) \in \Theta(\phi)$ if and only if there exist two finite sequences of terms $t_1(\overline{x}_1,x),\dots,t_n(\overline{x}_n,x)$ and pairs $(a_1,b_1),\dots,(a_n,b_n) \in \phi$ with*

$$a = t_1(\overline{x}_1,a_1),\ t_i(\overline{x}_i,b_i) = t_{i+1}(\overline{x}_{i+1},a_{i+1}) \text{ and } t_n(\overline{x}_n,b_n) = b$$

for some $\overline{x}_1,\dots,\overline{x}_n \in A$.

We say that an algebra $\mathbf{B}$ has the Congruence Extension Property (CEP for short) if, for any algebra $\mathbf{A}$ such that $\mathbf{B}$ is a subalgebra of $\mathbf{A}$ and for any congruence $\theta \in \mathrm{Con}\,\mathbf{B}$, there is a congruence $\phi \in \mathrm{Con}\,\mathbf{A}$ such that $\theta = (B \times B) \cap \phi$. A variety $\mathcal{K}$ has the CEP if every algebra in $\mathcal{K}$ has the CEP. For example, the variety of MV-algebra, or the variety of BL-algebras or the variety of state-morphism MV-algebras (see [13, Lem 6.1]) satisfies the CEP.

Theorem 4.7. *A variety $\mathcal{V}_\tau$ satisfy the CEP if and only if $\mathcal{V}$ satisfies the CEP.*

Proof. Let us have a variety $\mathcal{V}$ with the CEP. If $\mathbf{A} \in \mathcal{V}$ is such that $(\mathbf{A}, \tau)$ is an algebra with state-morphism, for any subalgebra $(\mathbf{B}, \tau) \subseteq (\mathbf{A}, \tau)$ and any $\phi \in \mathrm{Con}\,(\mathbf{B}, \tau)$, the condition $\phi = B^2 \cap \Theta(\phi)$ holds.

Now we prove $\Theta(\phi) = \Theta_\tau(\phi)$. To show that, assume $(a, b) \in \Theta(\phi)$. Mal'cev's Theorem shows the existence of finite sequences of terms $t_1(\overline{x}_1, x), \ldots, t_n(\overline{x}_n, x)$ and pairs $(a_1, b_1), \ldots, (a_n, b_n) \in \phi$ with

$$a = t_1(\overline{x}_1, a_1),\ t_i(\overline{x}_i, b_i) = t_{i+1}(\overline{x}_{i+1}, a_{i+1}) \text{ and } t_n(\overline{x}_n, b_n) = b$$

for some $\overline{x}_1, \ldots, \overline{x}_n \in A$. Because τ is an endomorphism, we obtain also equalities

$$\tau(a) = t_1(\tau(\overline{x}_1), \tau(a_1)),\ t_i(\tau(\overline{x}_i), \tau(b_i)) = t_{i+1}(\tau(\overline{x}_{i+1}), \tau(a_{i+1}))$$

and

$$t_n(\tau(\overline{x}_n), \tau(b_n)) = \tau(b).$$

We have assumed that $\phi \in \mathrm{Con}\,(\mathbf{B}, \tau)$, thus $(a_i, b_i) \in \phi$ yields $(\tau(a_i), \tau(b_i)) \in \phi$ for any $i = 1, \ldots, n$. Now, we have obtained $(\tau(a), \tau(b)) \in \Theta(\phi)$. In other words, $\Theta(\phi) \in \mathrm{Con}\,(\mathbf{A}, \tau)$ and thus $\Theta(\phi) = \Theta_\tau(\phi)$.

If $\mathcal{V}_\tau$ has the CEP, then for any $\mathbf{A} \in \mathcal{V}$, we have $\mathrm{Con}\,\mathbf{A} = \mathrm{Con}\,(\mathbf{A}, \mathrm{Id}_A)$. Clearly, the CEP on $(\mathbf{A}, \mathrm{Id}_A)$ yields the CEP on $\mathbf{A}$. □

5. Applications to Special Types of Algebras

In this section, we apply a general result concerning generators of some varieties of state-morphism algebras, Theorem 4.3, to the variety of state-morphism BL-algebras, state-morphism MTL-algebras, state-morphism non-associative BL-algebras, and state-morphism pseudo MV-algebras, when we use different systems of t-norms on the real interval $[0, 1]$ and a special type of pseudo MV-algebras, respectively.

Algebras for which the logic MTL is sound are called MTL-algebras. They can be characterized as prelinear commutative bounded integral residuated lattices. In more detail, according to [15], an algebraic structure $\mathbf{A} = (A; \wedge, \vee, *, \to, 0, 1)$ of type $\langle 2, 2, 2, 2, 0, 0\rangle$ is an *MTL-algebra* if

- (M1) $(A; \wedge, \vee, 0, 1)$ is a bounded lattice with the top element 0 and bottom element 1,
- (M2) $(A; *, 1)$ is a commutative monoid,
- (M3) $*$ and $\to$ form an adjoint pair, that is, $z * x \leq y$ if and only if $z \leq x \to y$, where $\leq$ is the lattice order of $(A; \wedge, \vee)$ for all $x, y, z \in A$, (the residuation condition),
- (M4) $(x \to y) \vee (y \to x) = 1$ holds for all $x, y \in A$ (the prelinearity condition).

If t is any left-continuous t-norm on $[0, 1]$, we define two binary operations $*_t$ $\to_t$ on $[0, 1]$ via $x *_t y = t(x, y)$ and $x \to_t y = \sup\{z \in [0, 1] : t(z, x) \leq y\}$ for $x, y \in [0, 1]$,

then $\mathbb{I}_t = ([0,1]; \min, \max, *_t, \to_t, 0, 1)$ is an example of an MTL-algebra. An MTL-algebra $\mathbb{I}_t$ is a BL-algebra iff t is continuous.

Due to [15], the class $\mathcal{T}_{lc}$, which denotes the system of all BL-algebras $\mathbb{I}_t$, where t is a left-continuous t-norm on the interval $[0,1]$, generates the variety of MTL-algebras. This result was strengthened in [27] who introduced the class of regular left-continuous t-norms which is strictly smaller than the class of left-continuous t-norms, but they generate the variety of MTL-algebras.

According to [1], we say that an algebra $\mathbf{A} = (A; \vee, \wedge, \cdot, \to, 0, 1)$ of type $\langle 2,2,2,2,0,0\rangle$ is a *non-associative BL-algebra* (naBL-algebra in short) if

(A1) $(A; \vee, \wedge, 0, 1)$ is a bounded lattice,
(A2) $(A; \cdot, 1)$ is a commutative groupoid with the neutral element 1,
(A3) any $x, y, z \in A$ satisfy $x \cdot y \leq z$ if and only if $x \leq y \to z$,
(A4) algebra satisfy the divisibility axiom $(x \cdot (x \to y) = x \wedge y)$,
(A5) algebra satisfy the α-prelinearity and β-prelinearity $(x \to y \vee \alpha^a_b(y \to x) = x \to y \vee \beta^a_b(y \to x) = 1)$, where $\alpha^a_b(x) = (a \cdot b) \to (a \cdot (b \cdot x))$ and $\beta^a_b(x) = b \to (a \to ((a \cdot b) \cdot x))$.

A function $t : [0,1] \times [0,1] \to [0,1]$ on the interval $[0,1]$ of reals is said to be a *non-associative* t-norm (nat-norm briefly) if

(nat1) $([0,1]; t, 1)$ is a commutative groupoid with the neutral element 1,
(nat2) t is continuous in the usual sense,
(nat3) if $x, y, z \in [0,1]$ are such that $x \leq y$, then $t(x,z) \leq t(y,z)$.

According to [1, Thm 5], for any nat-norm there is a unique binary operation $\to_t$ satisfying the adjointness condition, i.e. $t(x,y) \leq z$ if and only if $x \leq y \to_t z$. Moreover, an algebra $\mathbb{I}^{na}_t := ([0,1]; \min, \max, t, \to_t, 0, 1)$ is an naBL-algebra.

The class of all naBL-algebras is denoted by $na\mathcal{BL}$ and $na\mathcal{T}$ denotes the class of all naBL-algebras $\mathbb{I}^{na}_t$ for any non-associative t-norm. The main result on non-associative BL-algebras says that $na\mathcal{T}$ is the generating class for the variety $na\mathcal{BL}$, [1, Thm 8]:

Theorem 5.1. *There hods*

$$na\mathcal{BL} = \mathsf{IP_S SP_U}(na\mathcal{T}).$$

Finally, we recall that a noncommutative generalization of MV-algebras was introduced in [17] as *pseudo MV-algebras* or in [25] as *generalized MV-algebras*. According to [10], every pseudo MV-algebra $(M; \oplus, ^-, ^\sim, 0, 1)$ of type $\langle 2,1,1,0,0\rangle$ is an interval in a unital ℓ-group (G,u) with strong unit u, i.e. $M \cong \Gamma(G,u) := [0,u]$, where $x \oplus y = (x+y)\wedge$, $x^- = u - x$, $x^\sim = -x + u$, $0 = 0$, and $1 = u$. If (G,u) is double transitive (for definitions and details see [12]), then $\Gamma(G,u)$ generates the variety of pseudo MV-algebras, [12, Thm 4.8]. For example, if $\mathrm{Aut}(\mathbb{R})$ is the set of all automorphisms of the real line $\mathbb{R}$ preserving the natural order in $\mathbb{R}$ and $u(t) := t+1$, $t \in \mathbb{R}$, let $\mathrm{Aut}_u(\mathbb{R}) = \{g \in \mathrm{Aut}(\mathbb{R}) : g \leq nu$ for some integer $n \geq 1\}$. Then $\Gamma(\mathrm{Aut}_u(\mathbb{R}), u)$ is double transitive and it generates the variety of pseudo MV-algebras, see [12, Ex 5.3].

Now we apply the general statement, Theorem 4.4, on generators to different types of state-morphism algebras. We recall that $\mathcal{T}$ was defined as the class of all BL-algebras $\mathbb{I}_t$, where t is a continuous t-norm on $[0,1]$.

Theorem 5.2. (1) *The variety of all state-morphism MV-algebras is generated by the diagonal state-morphism MV-algebra* $D([0,1]_{MV})$.

(2) *The variety of all state-morphism BL-algebras is generated by the class* $\{D(\mathbb{I}_t) : \mathbb{I}_t \in \mathcal{T}\}$.

(3) *The variety of all state-morphism MTL-algebras is generated by the class* $\{D(\mathbb{I}_t) : \mathbb{I}_t \in \mathcal{T}_{lc}\}$.

(4) *The variety of all state-morphism naBL-algebras is generated by the class* $\{D(\mathbb{I}_t^{na}) : \mathbb{I}_t \in na\mathcal{T}\}$.

(5) *If a unital* ℓ*-group* (G, u) *is double transitive, then* $D(\Gamma(G, u))$ *generates the variety of state-morphism pseudo MV-algebras.*

Proof. (1) It follows from the fact that the MV-algebra of the real interval $[0, 1]$ generates the variety of MV-algebras, see e.g. [4, Prop 8.1.1], and then apply Theorem 4.4.

(2) The statement follows from the fact that $\mathsf{V}(\mathcal{T})$ is by [3, Thm 5.2] the variety $\mathcal{BL}$ of all BL-algebras. Now it suffices to apply Theorem 4.4.

(3) By [15], the class $\mathcal{T}_{lc}$ of all $\mathbb{I}_t$, where t is any left-continuous t-norms on the interval $[0, 1]$, generates the variety of MTL-algebras; then apply Theorem 4.4.

(4) By [1, Thm 8] or Theorem 5.1, the class $na\mathcal{T}$ of all $\mathbb{I}_t$, where t is any non-associative t-norms on the interval $[0, 1]$, generates the variety of non-associative BL-algebras; then apply again Theorem 4.4.

(5) By the above, $\Gamma(G, u)$ generates the variety of pseudo MV-algebras, see also [12, Thm 4.8]; then apply Theorem 4.4. □

We note that the case (1) in Theorem 4.4 was an open problem posed in [7] and was positively solved in [13, Thm 5.4(3)].

6. Conclusion

In the paper, we have presented a general approach to theory of state-morphism algebras which generalizes state-morphism MV-algebras and state-morphism BL-algebras as pairs $(\mathbf{A}, \tau)$, where $\mathbf{A}$ is an algebra of type F and τ is an endomorphism of $\mathbf{A}$ such that $\tau \circ \tau = \tau$.

This enables us to present complete characterizations of subdirectly irreducible state BL-algebras and subdirectly irreducible state-morphism BL-algebras, Theorem 2.7, which generalizes the results from [7, 9, 11, 13].

A general approach is studied in the third section where the main result, Theorem 3.7, says that every subdirectly irreducible state-morphism algebra can be embedded into a diagonal one.

The fourth section describes some generators of the varieties of state-morphism algebras, and Theorem 4.4 shows that if a class $\mathcal{K}$ generates a variety $\mathcal{V}$ of algebras of the same type F, then the variety of state-morphism algebras whose F-reduct belongs to the class $\mathcal{K}$ is generated by the class of diagonal state-morphism algebras $D(\mathbf{A})$, where $\mathbf{A} \in \mathcal{K}$. In addition, Theorem 4.7 deals with the CEP for the variety of state-morphism algebras.

In Theorem 5.2, Theorem 4.4 was applied to the special class of algebras: MV-algebras, BL-algebras, MTL-algebras, non-associative BL-algebras, and pseudo MV-algebras to obtain the generators of the corresponding varieties of state-morphism algebras.

During the study on this paper, we found some interesting open problems like: (1) find a characterization of an analogue of a state-operator that is not necessarily a state-morphism operator, (2) if the lattice of varieties of some variety is countable,

how big is the lattice of corresponding state-morphism algebras, e.g. in the case of MV-algebras, the lattice under question is uncountable [13], (3) decidability of the variety of state-morphism algebras.

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